

Near horizon black holes in diverse dimensions and integrable models

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Abstract

The near horizon geometry of an extremal rotating black hole in arbitrary dimension possesses $SO(2,1) \times U(n)$ symmetry in the special case that all n rotation parameters are equal. We consider a conformal particle associated with such a maximally symmetric configuration and derive from it a new integrable Hamiltonian mechanics with $U(n)$ symmetry. A further reduction of the model is discussed, which is obtained by discarding cyclic variables. A variant of the Higgs oscillator and the Pöschl–Teller system show up in the case of four and five dimensions, respectively.

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1. Introduction

In recent years there has been considerable interest in various aspects of the Kerr/CFT correspondence [1].¹ To a large extent the original proposal in [1] was motivated by the earlier work [4], where it was demonstrated that the isometry group $U(1) \times U(1)$ of the extremal Kerr black hole in four dimensions was enlarged to $SO(2, 1) \times U(1)$ in the near horizon limit. Note that the first factor is the conformal group in one dimension.

In the conventional formulation of the Kerr/CFT correspondence [1], one considers excitations around the near horizon extremal Kerr black hole, which are controlled by specific boundary conditions. For every set of boundary conditions there is an associated asymptotic symmetry group formed by diffeomorphisms consistent with the conditions. A key ingredient of the construction is a computation of conserved charges related to the asymptotic symmetry transformations. An asymptotic symmetry is considered to be trivial if the corresponding charge vanishes. A curious fact about the Kerr/CFT correspondence is that the $U(1)$ factor in the isometry group of the Kerr metric is enhanced to a chiral Virasoro algebra, while $SO(2, 1)$ becomes trivial.

As $SO(2, 1)$ symmetry of the near horizon extremal Kerr solution appears to play a trivial role within the Kerr/CFT correspondence, it is worth studying its implication in a different context. In this work we employ the conformal symmetry of the near horizon extremal rotating black hole in arbitrary dimension to construct new (super)integrable many-body models.

As is well known, each Killing vector field characterizing a background geometry yields an integral of motion of the geodesic equation. In particular, a massive relativistic particle moving near the horizon of an extremal rotating black hole in arbitrary dimension is conformal invariant.² Recently it was demonstrated [16, 17] that the Casimir element of $so(2, 1)$ algebra underlying a generic conformal mechanics gives rise to a reduced Hamiltonian system called a spherical mechanics. In this work we initiate a systematic study of a spherical mechanics associated with the extremal rotating black hole in arbitrary dimension and derive its Hamiltonian.

In arbitrary dimension, a black hole may rotate in various orthogonal spatial two-planes. Given n independent rotation parameters, the isometry group of the metric involves $U(1)^n$ factor. If all the rotation parameters be equal, the isometry group is enhanced to $U(n)$. In this work we focus on such a maximally symmetric configuration and construct new (super)integrable models, which inherit $U(n)$ symmetry of the background.

In the next section we discuss the near horizon limit of the extremal Kerr black hole in four dimensions. Conformal mechanics associated to it is constructed and a reduced two-dimensional integrable spherical mechanics is derived. A further reduction of the latter to a one-dimensional system, which is obtained by discarding cyclic variables, is shown to yield

¹By now there is a rather extensive literature on the subject. For recent reviews and further references to the original literature see e.g. [2, 3].

²For recent studies of conformal mechanics related to the near horizon geometry of extremal black holes see e.g. [5]–[15].

a variant of the Higgs oscillator [18]. In Section 3 we consider the near horizon limit of the five-dimensional Myers–Perry black hole. The corresponding conformal mechanics is described and a three-dimensional reduced integrable spherical mechanics is constructed. If the rotation parameters of the black hole be equal to each other, the resulting model is shown to be minimally superintegrable. A further reduction to a one-dimensional system, which is obtained by discarding cyclic variables, is shown to yield the Pöschl–Teller system [19]. In Section 4 we extend the analysis to the case of an extremal rotating black hole in $d = 2n$. The near horizon limit is constructed for a spacial case that all n rotation parameters are equal. The associated conformal mechanics is analyzed and the Hamiltonian of a $(2n - 2)$ -dimensional reduced integrable spherical mechanics is given. Section 5 contains similar results for $d = 2n + 1$. In the concluding Section 6 we summarize our results and discuss possible further developments.

2. $2d$ integrable model related to $4d$ Kerr black hole

2.1. Near horizon geometry of $4d$ extremal Kerr black hole

In Boyer–Lindquist-type coordinates the Kerr solution of the Einstein equations reads³

$$ds^2 = dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \frac{2Mr}{\rho^2} (dt - a \sin^2 \theta d\phi)^2 - (r^2 + a^2) \sin^2 \theta d\phi^2, \\ \Delta = r^2 + a^2 - 2Mr, \quad \rho^2 = r^2 + a^2 \cos^2 \theta, \quad (1)$$

where M is the mass and a is the rotation parameter. The isometry group of (1) includes the time translation and the rotation around z -axis

$$t' = t + \alpha, \quad \phi' = \phi + \beta. \quad (2)$$

A salient feature of the Kerr geometry is that it possesses a hidden symmetry described by the second rank Killing tensor $(x^m = (t, r, \theta, \phi))$ [20, 21, 22]

$$K_{mn} = Q_{mn} + r^2 g_{mn}, \quad \nabla_{(n} K_{mp)} = 0, \quad (3)$$

where

$$Q_{mn} = \begin{pmatrix} -\Delta & 0 & 0 & a\Delta \sin^2 \theta \\ 0 & \frac{\rho^4}{\Delta} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a\Delta \sin^2 \theta & 0 & 0 & -a^2 \Delta \sin^4 \theta \end{pmatrix}.$$

³Throughout the paper we use the mostly minus signature convention for the metric tensor and put $c = 1$, $G = 1$.

In particular, it renders the geodesic motion integrable.

Zeros of Δ determine the inner and outer horizons which coalesce for the extremal solution. Denoting the corresponding value of the radial coordinate by r_0 and assuming a to be positive, from the equations $\Delta(r_0) = 0$, $\Delta'(r_0) = 0$ one finds

$$r_0 = M = a. \quad (4)$$

In what follows we discuss the extremal solution only.

The construction of the near horizon Kerr geometry is tricky and it deserves to be discussed in detail. A natural definition of the near horizon limit, which implies the redefinition of the radial coordinate

$$r \rightarrow r_0 + \epsilon r_0 r, \quad (5)$$

followed by $\epsilon \rightarrow 0$, yields a degenerate metric. Yet, an important observation is that (5) transforms the radial term $\frac{\rho^2}{\Delta} dr^2$ into $r_0^2(1 + \cos^2 \theta) \frac{dr^2}{r^2}$, the last factor of which is a part of the AdS_2 metric $r^2 dt^2 - \frac{dr^2}{r^2}$. This prompts one to extract $r^2 dt^2$ from the fourth term in the metric (1) by a proper redefinition of the temporal coordinate t and the azimuthal angle ϕ [4]. To be more precise, one rewrites the term as

$$\frac{\Delta}{\rho^2} (dt - a \sin^2 \theta d\phi)^2 - \frac{(r^2 + a^2)}{\rho^2} (dt - a \sin^2 \theta d\phi)^2, \quad (6)$$

and treats the two contributions separately. The first of them yields the desired expression $r_0^2(1 + \cos^2 \theta) r^2 dt^2$ provided (5) is extended by the new prescriptions

$$t \rightarrow \frac{2r_0 t}{\epsilon}, \quad \phi \rightarrow \phi + \frac{t}{\epsilon}. \quad (7)$$

The second term in (6) is then combined with the rest in (1) involving dt and $d\phi$ so as to form the expression $-\frac{\sin^2 \theta}{\rho^2} (adt - (r^2 + a^2)d\phi)^2$, which behaves well in the limit (5), (7). Combining all the pieces together, one gets the extremal Kerr throat solution [4]

$$ds^2 = (1 + \cos^2 \theta) \left(r^2 dt^2 - \frac{dr^2}{r^2} - d\theta^2 \right) - \frac{4 \sin^2 \theta}{(1 + \cos^2 \theta)} (r dt + d\phi)^2, \quad (8)$$

where the overall factor of r_0^2 has been discarded.⁴ It is amazing that the near horizon solution does not involve any physical parameters.

Near the throat the isometry group is enhanced. In addition to (2) it includes the dilatation

$$t' = t + \gamma t, \quad r' = r - \gamma r, \quad (9)$$

and the special conformal transformation

$$t' = t + (t^2 + \frac{1}{r^2})\sigma, \quad r' = r - 2tr\sigma, \quad \phi' = \phi - \frac{2}{r}\sigma. \quad (10)$$

⁴Note that in different coordinates this metric was derived earlier in [23]. Conformal symmetry of the solution was not discussed in [23], however.

Altogether they form $SO(2, 1) \times U(1)$ group.

The recipe (5), (7) can also be used to derive the second rank Killing tensor in the near horizon region

$$K_{nm}dx^n dx^m = (1 + \cos^2 \theta)^2 \left[r^2 dt^2 - \frac{1}{r^2} dr^2 \right], \quad (11)$$

where a contribution proportional to the metric was discarded. However, because the isometry group of the near horizon metric is enhanced, the Killing tensor proves to be reducible [10]

$$K_{nm} = \frac{1}{2} (\xi_n^{(1)} \xi_m^{(3)} + \xi_n^{(3)} \xi_m^{(1)}) - \xi_n^{(2)} \xi_m^{(2)} + \xi_n^{(4)} \xi_m^{(4)}. \quad (12)$$

Here $(\xi_n^{(1)}, \xi_n^{(2)}, \xi_n^{(3)}, \xi_n^{(4)})$ denote the Killing vectors corresponding to the time translation, dilatation, special conformal transformation and rotation around z -axis, respectively. As usual, the index is lowered with the use of the metric.

2.2. 4d conformal mechanics and its integrable reductions

The static gauge action functional for a massive particle propagating near the horizon of the 4d extremal Kerr black hole reads

$$S = -m \int dt \sqrt{(1 + \cos^2 \theta) \left[r^2 - \dot{r}^2/r^2 - \dot{\theta}^2 \right] - \frac{4 \sin^2 \theta}{(1 + \cos^2 \theta)} \left[r + \dot{\phi} \right]^2}, \quad (13)$$

where the dot denotes the derivative with respect to t . We choose the Hamiltonian formalism to analyze the model. Introducing momenta (p_r, p_θ, p_ϕ) canonically conjugate to the configuration space variables (r, θ, ϕ) , one can readily derive the Hamiltonian and the generators of special conformal transformations and dilatations from the Killing vector fields (2), (9), (10). The general formula which links a Killing vector field with the components $\xi^n(x)$ to the first integral $\xi^n(x)g_{nm}(x)\frac{dx^n}{ds}$ of the geodesic equation yields

$$H = r \left(\sqrt{m^2(1 + \cos^2 \theta) + (rp_r)^2 + p_\theta^2 + \left(\frac{1 + \cos^2 \theta}{2 \sin \theta} \right)^2 p_\phi^2} - p_\phi \right), \quad D = tH + rp_r, \\ K = \frac{1}{r} \left(\sqrt{m^2(1 + \cos^2 \theta) + (rp_r)^2 + p_\theta^2 + \left(\frac{1 + \cos^2 \theta}{2 \sin \theta} \right)^2 p_\phi^2} + p_\phi \right) + t^2 H + 2trp_r. \quad (14)$$

Under the Poisson bracket the functions obey the structure relations of $so(2, 1)$ algebra

$$\{H, D\} = H, \quad \{H, K\} = 2D, \quad \{D, K\} = K. \quad (15)$$

The Killing vector corresponding to the rotation symmetry leads to the conserved momentum p_ϕ , while the Killing tensor (11) gives

$$m^2(1 + \cos^2 \theta) + p_\theta^2 + \left(\frac{1 + \cos^2 \theta}{2 \sin \theta} \right)^2 p_\phi^2 = HK - D^2 + p_\phi^2. \quad (16)$$

Thus, in the near horizon region the second rank Killing tensor reduces to a combination of the Casimir element of $so(2, 1)$ and p_ϕ^2 [10, 11]. Note that the reduction formula (12) is most easily derived from (16).

Like for a generic conformal mechanics [16, 17], the Casimir element of $so(2, 1)$ realized in the model of a massive relativistic particle propagating near the horizon of an extremal black hole gives rise to a reduced Hamiltonian system [13]. For the case at hand one finds a two-dimensional system governed by the Hamiltonian

$$\mathcal{H} = p_\theta^2 + \left(\left[\frac{1 + \cos^2 \theta}{2 \sin \theta} \right]^2 - 1 \right) p_\phi^2 + m^2 (1 + \cos^2 \theta), \quad (17)$$

where m is now treated as a coupling constant. As p_ϕ commutes with \mathcal{H} , the system is Liouville integrable. For generic values of the parameters m and p_ϕ a solution of the canonical equations of motion, which follow from (17), involves elliptic integrals [10, 14].

Because ϕ is cyclic, it is worth considering a further reduction of (17) to a one-dimensional system, which is obtained by setting the conserved momentum p_ϕ to be a coupling constant directly in the Hamiltonian

$$\tilde{\mathcal{H}}_{red} = p_\theta^2 + g^2 \cot^2 \theta + \nu \cos^2 \theta. \quad (18)$$

Here g and ν are coupling constants related to the parameters of the original particle via $g^2 = p_\phi^2$ and $\nu = m^2 - \frac{1}{4}p_\phi^2$. As usual, the Hamiltonian is defined up to an additive constant. The resulting system (18) is a variant of the celebrated Higgs oscillator [18] coupled to an external field. It is important to stress that, in view of (16), the Hamiltonian of the reduced spherical mechanics is linked to the near horizon Killing tensor.

3. 3d integrable model related to 5d Myers–Perry black hole

3.1. Near horizon geometry of the extremal Myers–Perry black hole

A generalization of the Kerr solution to the case of a five-dimensional spacetime was constructed by Myers and Perry [24]

$$\begin{aligned} ds^2 &= dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \frac{2M}{\rho^2} (dt - a \sin^2 \theta d\phi - b \cos^2 \theta d\psi)^2 - \\ &\quad - (r^2 + a^2) \sin^2 \theta d\phi^2 - (r^2 + b^2) \cos^2 \theta d\psi^2, \\ \Delta &= \frac{1}{r^2} (r^2 + a^2)(r^2 + b^2) - 2M, \quad \rho^2 = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \end{aligned} \quad (19)$$

where M is the mass and a, b are rotation parameters. The isometry group of (19) is $U(1)^3$, which includes the time translation and two rotations

$$t' = t + \alpha, \quad \phi' = \phi + \beta, \quad \psi' = \psi + \gamma. \quad (20)$$

The explicit form of the second rank Killing tensor can be found in [25]. Note that, if the rotation parameters a and b are equal to each other, the isometry group of the Myers–Perry metric is enhanced to $U(2) \times U(1)$ [26]. The extra symmetries read

$$\begin{aligned}\theta' &= \theta + \mu \cos(\psi - \phi), & \theta' &= \theta - \nu \sin(\psi - \phi), \\ \phi' &= \phi + \mu \cot \theta \sin(\psi - \phi), & \phi' &= \phi + \nu \cot \theta \cos(\psi - \phi), \\ \psi' &= \psi + \mu \tan \theta \sin(\psi - \phi), & \psi' &= \psi + \nu \tan \theta \cos(\psi - \phi),\end{aligned}\tag{21}$$

where μ and ν are infinitesimal parameters.

The extremal solution occurs if Δ has a double zero at the horizon radius r_0 . Assuming a and b to be positive, from $\Delta(r_0) = 0$ and $\Delta'(r_0) = 0$ one finds

$$r_0^2 = ab, \quad M = \frac{(a+b)^2}{2}.\tag{22}$$

In what follows we will also need the relation

$$\lim_{r \rightarrow r_0} \frac{\Delta}{(r - r_0)^2} = 4.\tag{23}$$

The near horizon limit of the Myers–Perry solution is constructed by analogy with the four dimensional case. One considers the second and the fourth terms in the metric (19) and rewrites them in the form

$$-\frac{\rho^2}{\Delta} dr^2 + \frac{\Delta}{\rho^2} (dt - a \sin^2 \theta d\phi - b \cos^2 \theta d\psi)^2 - \frac{(r^2 + a^2)(r^2 + b^2)}{r^2 \rho^2} (dt - a \sin^2 \theta d\phi - b \cos^2 \theta d\psi)^2.\tag{24}$$

The first two contributions entering (24) are reserved to produce the AdS_2 metric up to a factor. The last term in (24) and the rest in (19) are combined together to form

$$\begin{aligned}& -\rho^2 d\theta^2 - \frac{\sin^2 \theta}{\rho^2} [adt - (r^2 + a^2)d\phi]^2 - \frac{\cos^2 \theta}{\rho^2} [bdt - (r^2 + b^2)d\psi]^2 \\ & - \frac{1}{r^2 \rho^2} [abdt - b(r^2 + a^2) \sin^2 \theta d\phi - a(r^2 + b^2) \cos^2 \theta d\psi]^2.\end{aligned}\tag{25}$$

At this stage one redefines the coordinates

$$r \rightarrow r_0 + \epsilon r_0 r, \quad t \rightarrow \frac{\alpha t}{\epsilon}, \quad \phi \rightarrow \phi + \frac{\beta t}{\epsilon}, \quad \psi \rightarrow \psi + \frac{\gamma t}{\epsilon},\tag{26}$$

and adjust the number coefficients α, β, γ so as to produce a finite result as $\epsilon \rightarrow 0$. Choosing

$$\alpha = \frac{(a+b)^2}{4r_0}, \quad \beta = \gamma = \frac{(a+b)}{4r_0},\tag{27}$$

and making use of (23), one finally gets [27]

$$\begin{aligned}
ds^2 = & \rho_0^2 \left[r^2 dt^2 - \frac{dr^2}{r^2} - 4d\theta^2 \right] - \frac{ab(a+b)^2 \sin^2 \theta}{\rho_0^2} [rdt + d\phi]^2 \\
& - \frac{ab(a+b)^2 \cos^2 \theta}{\rho_0^2} [rdt + d\psi]^2 - \frac{1}{\rho_0^2} [\rho_0^2 r dt + b(a+b) \sin^2 \theta d\phi + a(a+b) \cos^2 \theta d\psi]^2, \\
\rho_0^2 = & ab + a^2 \cos^2 \theta + b^2 \sin^2 \theta,
\end{aligned} \tag{28}$$

where for simplicity we discarded the overall factor of $1/4$ and scaled $\frac{2a}{r_0}\phi \rightarrow \phi$, $\frac{2b}{r_0}\psi \rightarrow \psi$.

The near horizon metric has a larger symmetry. In addition to (20), the isometry group of (28) includes the dilatation

$$t' = t + \lambda t, \quad r' = r - \lambda r, \tag{29}$$

and the special conformal transformation

$$t' = t + (t^2 + \frac{1}{r^2})\sigma, \quad r' = r - 2tr\sigma, \quad \phi' = \phi - \frac{2}{r}\sigma, \quad \psi' = \psi - \frac{2}{r}\sigma, \tag{30}$$

which altogether form $SO(2,1) \times U(1)^2$. For equal rotation parameters this is further extended to $SO(2,1) \times U(2)$. The extra symmetries can be realized as in (21), provided one scales the azimuthal coordinates entering the near horizon metric as follows: $\phi \rightarrow 2\phi$, $\psi \rightarrow 2\psi$.

Like in four dimensions, the near horizon Killing tensor

$$K_{nm} dx^n dx^m = \rho_0^4 \left[r^2 dt^2 - \frac{1}{r^2} dr^2 \right], \tag{31}$$

reduces to a quadratic combination of the Killing vectors

$$K_{nm} = \frac{1}{2} (\xi_n^{(1)} \xi_m^{(3)} + \xi_n^{(3)} \xi_m^{(1)}) - \xi_n^{(2)} \xi_m^{(2)} + \xi_n^{(4)} \xi_m^{(4)} + \xi_n^{(5)} \xi_m^{(5)} + (\xi_n^{(4)} \xi_m^{(5)} + \xi_n^{(5)} \xi_m^{(4)}), \tag{32}$$

where $(\xi_n^{(1)}, \xi_n^{(2)}, \xi_n^{(3)}, \xi_n^{(4)}, \xi_n^{(5)})$ denote the Killing vectors corresponding to the time translation, dilatation, special conformal transformation, and the shift of ϕ and ψ , respectively.

3.2. 5d conformal mechanics and its integrable reductions

Given the near horizon metric (28), the static gauge action functional of a massive particle propagating on this background reads

$$\begin{aligned}
S = & -m \int dt \left[\rho_0^2 \left(r^2 - \frac{\dot{r}^2}{r^2} - 4\dot{\theta}^2 \right) - \frac{ab(a+b)^2}{\rho_0^2} \left(\sin^2 \theta (r + \dot{\phi})^2 + \cos^2 \theta (r + \dot{\psi})^2 \right) \right. \\
& \left. - \frac{1}{\rho_0^2} \left(r \rho_0^2 + b(a+b) \sin^2 \theta \dot{\phi} + a(a+b) \cos^2 \theta \dot{\psi} \right)^2 \right]^{1/2},
\end{aligned} \tag{33}$$

where $\rho_0^2 = ab + a^2 \cos^2 \theta + b^2 \sin^2 \theta$ and m is the particle mass. Introducing momenta $(p_r, p_\theta, p_\phi, p_\psi)$ canonically conjugate to the configuration space variables (r, θ, ϕ, ψ) and taking into account the explicit form of the Killing vectors specified in the preceding section, one derives the Hamiltonian and the generators of dilatations and special conformal transformations

$$\begin{aligned} H &= r \sqrt{m^2 \rho_0^2 + (rp_r)^2 + \frac{1}{4} p_\theta^2 + \frac{\rho_0^4}{ab(a+b)^2} \left(\frac{p_\phi^2}{\sin^2 \theta} + \frac{p_\psi^2}{\cos^2 \theta} - \frac{1}{\rho_0^2} (bp_\phi + ap_\psi)^2 \right)} - \\ &\quad - r(p_\phi + p_\psi), \\ K &= \frac{1}{r} \sqrt{m^2 \rho_0^2 + (rp_r)^2 + \frac{1}{4} p_\theta^2 + \frac{\rho_0^4}{ab(a+b)^2} \left(\frac{p_\phi^2}{\sin^2 \theta} + \frac{p_\psi^2}{\cos^2 \theta} - \frac{1}{\rho_0^2} (bp_\phi + ap_\psi)^2 \right)} + \\ &\quad + \frac{1}{r} (p_\phi + p_\psi) + t^2 H + 2trp_r, \quad D = tH + rpr. \end{aligned} \quad (34)$$

The Killing vectors corresponding to rotations reproduce the conserved momenta p_ϕ and p_ψ , while the Killing tensor (31) gives

$$m^2 \rho_0^2 + \frac{1}{4} p_\theta^2 + \frac{\rho_0^4}{ab(a+b)^2} \left(\frac{p_\phi^2}{\sin^2 \theta} + \frac{p_\psi^2}{\cos^2 \theta} - \frac{1}{\rho_0^2} (bp_\phi + ap_\psi)^2 \right) = HK - D^2 + (p_\phi + p_\psi)^2. \quad (35)$$

The right hand side of (35) involves the Casimir element of $so(2,1)$ which specifies the Hamiltonian of a reduced $3d$ system

$$\mathcal{H} = \frac{1}{4} p_\theta^2 + \frac{\rho_0^4}{ab(a+b)^2} \left(\frac{p_\phi^2}{\sin^2 \theta} + \frac{p_\psi^2}{\cos^2 \theta} - \frac{1}{\rho_0^2} (bp_\phi + ap_\psi)^2 \right) - (p_\phi + p_\psi)^2 + m^2 \rho_0^2, \quad (36)$$

where $\rho_0^2 = ab + a^2 \cos^2 \theta + b^2 \sin^2 \theta$. As p_ϕ and p_ψ commute with the Hamiltonian, the system is Liouville integrable.

If the rotation parameters a and b of the original black hole are equal to each other, the Hamiltonian (36) simplifies to

$$\tilde{\mathcal{H}} = \frac{1}{4} p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta} + \frac{p_\psi^2}{\cos^2 \theta} - \frac{3}{2} (p_\phi + p_\psi)^2. \quad (37)$$

In accord with our consideration above, the model possesses four integrals of motion, which are linked to $U(2)$ factor in the isometry group

$$\begin{aligned} J_0 &= p_\psi + p_\phi, & J_1 &= p_\psi - p_\phi, \\ J_2 &= \frac{1}{2} p_\theta \cos \left(\frac{1}{2} (\psi - \phi) \right) + (p_\phi \cot \theta + p_\psi \tan \theta) \sin \left(\frac{1}{2} (\psi - \phi) \right), \\ J_3 &= \frac{1}{2} p_\theta \sin \left(\frac{1}{2} (\psi - \phi) \right) - (p_\phi \cot \theta + p_\psi \tan \theta) \cos \left(\frac{1}{2} (\psi - \phi) \right). \end{aligned} \quad (38)$$

In particular, under the Poisson bracket J_1 , J_2 and J_3 obey the structure relations of $su(2)$ and commute with J_0 . The Hamiltonian is proportional to the Casimir element of $su(2)$

$$\tilde{\mathcal{H}} = J_i J_i - \frac{3}{2} J_0^2. \quad (39)$$

As there are four functionally independent integrals of motion in a system with three degrees of freedom, (37) determines a minimally superintegrable model.

Because the variables ϕ and ψ are cyclic, one might be interested in a further reduction of (37) to a one-dimensional system. Setting in (37) the momenta p_ϕ and p_ψ to be coupling constants, one gets

$$\tilde{\mathcal{H}}_{red} = \frac{1}{4} p_\theta^2 + \frac{\nu_1^2}{\sin^2 \theta} + \frac{\nu_2^2}{\cos^2 \theta}. \quad (40)$$

This is a variant of the celebrated Pöschl–Teller model [19].

4. Extremal black holes in $d = 2n$ and integrable models

4.1. Near horizon geometry of the extremal maximally symmetric $d = 2n$ black hole

A generalization of the Kerr solution of the Einstein equations to the case of even-dimensional space-time was proposed in [24]. In Boyer–Lindquist-type coordinates it reads

$$ds^2 = dt^2 - \frac{U}{\Delta} dr^2 - \frac{2M}{U} \left(dt - \sum_{i=1}^{n-1} a_i \mu_i^2 d\phi_i \right)^2 - \sum_{i=1}^n (r^2 + a_i^2) d\mu_i^2 - \sum_{i=1}^{n-1} (r^2 + a_i^2) \mu_i^2 d\phi_i^2, \\ \Delta = \frac{1}{r} \prod_{i=1}^{n-1} (r^2 + a_i^2) - 2M, \quad U = r \sum_{i=1}^n \frac{\mu_i^2}{r^2 + a_i^2} \prod_{j=1}^{n-1} (r^2 + a_j^2), \quad (41)$$

where the latitudinal coordinates μ_i obey the constraint

$$\sum_{i=1}^n \mu_i^2 = 1. \quad (42)$$

It is assumed that only $(n - 1)$ independent rotation parameters are present so a_n is set to zero in (41). The range of azimuthal coordinates ϕ_i is taken to be $[0, 2\pi]$, μ_i lie in the interval $[0, 1]$ for $i = 1, \dots, n - 1$, while $\mu_n \in [-1, 1]$. The isometry group of (41) includes the time translation and $(n - 1)$ rotations which altogether form $U(1)^n$.

Because in this work we are primarily concerned with the construction of superintegrable systems, in what follows we consider only the special case that all the rotation parameters are equal $a_i = a$. In this case the $U(1)^{n-1}$ subgroup in the isometry group, which corresponds to rotations, is known to enhance to $U(n - 1)$ (see e.g. the discussion in [28]). Integrable

models which follow from less symmetric configurations will be studied elsewhere. Note that for unequal constants a_i the near horizon metric has been constructed in [27].

Before implementing the near horizon limit one has to put the metric in a convenient form. Following the steps outlined in the preceding sections, one finds

$$ds^2 = \frac{\Delta}{U} \left(dt - a \sum_{i=1}^{n-1} \mu_i^2 d\phi_i \right)^2 - \frac{U}{\Delta} dr^2 - \frac{(r^2 + a^2)^{n-2}}{rU} \sum_{i=1}^{n-1} \mu_i^2 (adt - (r^2 + a^2)d\phi_i)^2 \\ - (r^2 + a^2) \sum_{i=1}^{n-1} d\mu_i^2 - r^2 d\mu_n^2 + \frac{a^2(r^2 + a^2)^{n-1}}{rU} \sum_{i<j}^{n-1} \mu_i^2 \mu_j^2 (d\phi_i - d\phi_j)^2, \quad (43)$$

$$\Delta = \frac{1}{r} (r^2 + a^2)^{n-1} - 2M, \quad U = \frac{1}{r} (r^2 + a^2)^{n-2} (r^2 + a^2 \mu_n^2), \quad \mu_n^2 = 1 - \sum_{i=1}^{n-1} \mu_i^2.$$

The extremal solution is characterized by the condition that Δ has a double zero at the horizon radius $r = r_0$. The conditions $\Delta(r_0) = 0$ and $\Delta'(r_0) = 0$ allow one to fix the mass M and the rotation parameter a in terms of r_0

$$M = \frac{r_0^{2n-3} [2(n-1)]^{n-1}}{2}, \quad a^2 = (2n-3)r_0^2. \quad (44)$$

In what follows we will also need the relation

$$\lim_{r \rightarrow r_0} \frac{\Delta}{(r - r_0)^2} = \frac{(2n-3)[2(n-1)r_0^2]^{n-2}}{r_0}. \quad (45)$$

In order to implement the near horizon limit, one redefines the coordinates

$$r \rightarrow r_0 + \epsilon r_0 r, \quad t \rightarrow \frac{\alpha t}{\epsilon}, \quad \phi_i \rightarrow \phi_i + \frac{\beta_i t}{\epsilon}, \quad (46)$$

and adjusts the number coefficients α and β_i in such a way that the first two terms in (43) produce the AdS_2 metric (up to a factor) for $\epsilon \rightarrow 0$

$$\alpha = \frac{2(n-1)r_0}{2n-3}, \quad \beta_i = \frac{r_0}{a}. \quad (47)$$

Sending ϵ to zero and taking into account (45), one finally derives the near horizon metric

$$ds^2 = \rho_0^2 \left(r^2 dt^2 - \frac{dr^2}{r^2} \right) - 2(n-1) \sum_{i=1}^{n-1} d\mu_i^2 - d\mu_n^2 - \frac{4}{(2n-3)^2 \rho_0^2} \sum_{i=1}^{n-1} \mu_i^2 (r dt + d\phi_i)^2 + \\ + \frac{2}{(n-1)(2n-3)\rho_0^2} \sum_{i<j}^{n-1} \mu_i^2 \mu_j^2 (d\phi_i - d\phi_j)^2, \quad (48)$$

$$\rho_0^2 = \frac{1 + (2n-3)\mu_n^2}{2n-3}, \quad \mu_n^2 = 1 - \sum_{i=1}^{n-1} \mu_i^2,$$

where we discarded an overall factor of r_0^2 and scaled the azimuthal angular variables as follows: $\frac{a(n-1)}{r_0}\phi_i \rightarrow \phi_i$. It is straightforward to verify that (48) is a vacuum solution of the Einstein equations. Note that setting $n = 2$, $\mu_1 = \sin \theta$, $\mu_2 = \cos \theta$ (with $\theta \in [0, \pi]$) one reproduces the Bardeen–Horowitz metric (8).

Similarly to the lower dimensional patterns considered above, (48) exhibits conformal symmetry, which is realized as in Eqs. (9) and (10) with the obvious alteration of the special conformal transformation acting on the azimuthal angular variables

$$\phi'_i = \phi_i - \frac{2}{r}\sigma. \quad (49)$$

Killing vectors corresponding to the conformal transformations along with those generating the time translation and the shifts of the azimuthal angular coordinates can be used to demonstrate that the near horizon Killing tensor

$$K_{nm}dx^n dx^m = \rho_0^4 \left[r^2 dt^2 - \frac{1}{r^2} dr^2 \right], \quad (50)$$

with ρ_0 defined in (48), is reducible

$$K_{nm} = \frac{1}{2} (\xi_n^{(1)} \xi_m^{(3)} + \xi_n^{(3)} \xi_m^{(1)}) - \xi_n^{(2)} \xi_m^{(2)} + \tilde{\xi}_n \tilde{\xi}_m. \quad (51)$$

Here $(\xi_n^{(1)}, \xi_n^{(2)}, \xi_n^{(3)})$ denote the Killing vectors corresponding to the time translation, dilatation and special conformal transformation, respectively, while $\tilde{\xi}_n$ designates the sum of the Killing vectors related to the shifts of the azimuthal angular coordinates.

4.2. Conformal mechanics in $d = 2n$ and its integrable reductions

The construction of the Hamiltonian of a massive relativistic particle propagating on the curved background (48) is a straightforward, albeit somewhat tedious task. An alternative method is to invert the metric (48)

$$\begin{aligned} g^{mn}(x) \partial_n \partial_m &= \frac{1}{r^2 \rho_0^2} \left(\frac{\partial}{\partial t} \right)^2 - \frac{r^2}{\rho_0^2} \left(\frac{\partial}{\partial r} \right)^2 - \frac{2}{r \rho_0^2} \sum_{i=1}^{n-1} \frac{\partial}{\partial t} \frac{\partial}{\partial \phi_i} + \\ &+ \frac{1}{(2n-3)(2n-2)\rho_0^2} \sum_{i,j=1}^{n-1} (\mu_i \mu_j - (2n-3)\rho_0^2 \delta_{ij}) \frac{\partial}{\partial \mu_i} \frac{\partial}{\partial \mu_j} + \\ &+ \sum_{i,j=1}^{n-1} \left(\frac{(2n-3)^2}{4} + \frac{1}{\rho_0^2} - \frac{(2n-3)(2n-2)}{4\mu_i^2} \delta_{ij} \right) \frac{\partial}{\partial \phi_i} \frac{\partial}{\partial \phi_j}, \\ \rho_0^2 &= \frac{2(n-1)}{2n-3} - \sum_{i=1}^{n-1} \mu_i^2, \end{aligned} \quad (52)$$

where δ_{ij} is the Kronecker delta, and then solve the mass shell condition

$$g^{nm}p_n p_m = m^2, \quad (53)$$

where $p_m = (p_0, p_r, p_{\mu_i}, p_{\phi_i})$, for p_0 ⁵

$$p_0 = H = r \left(\left[m^2 \rho_0^2 + (r p_r)^2 - \frac{1}{(2n-3)(2n-2)} \sum_{i,j=1}^{n-1} (\mu_i \mu_j - (2n-3) \rho_0^2 \delta_{ij}) p_{\mu_i} p_{\mu_j} \right. \right. \\ \left. \left. - \sum_{i,j=1}^{n-1} \left(1 + \frac{(2n-3)^2 \rho_0^2}{4} - \frac{(2n-3)(2n-2) \rho_0^2}{4 \mu_i^2} \delta_{ij} \right) p_{\phi_i} p_{\phi_j} + \left(\sum_{i=1}^{n-1} p_{\phi_i} \right)^2 \right]^{1/2} - \sum_{i=1}^{n-1} p_{\phi_i} \right). \quad (54)$$

This gives the Hamiltonian of a conformal mechanics in $d = 2n$. Conformal generators are constructed by analogy with the lower dimensional patterns considered above

$$K = \frac{1}{r} \left(\left[m^2 \rho_0^2 + (r p_r)^2 - \frac{1}{(2n-3)(2n-2)} \sum_{i,j=1}^{n-1} (\mu_i \mu_j - (2n-3) \rho_0^2 \delta_{ij}) p_{\mu_i} p_{\mu_j} - \right. \right. \\ \left. \left. - \sum_{i,j=1}^{n-1} \left(1 + \frac{(2n-3)^2 \rho_0^2}{4} - \frac{(2n-3)(2n-2) \rho_0^2}{4 \mu_i^2} \delta_{ij} \right) p_{\phi_i} p_{\phi_j} + \left(\sum_{i=1}^{n-1} p_{\phi_i} \right)^2 \right]^{1/2} + \sum_{i=1}^{n-1} p_{\phi_i} \right) \\ + t^2 H + 2 t r p_r, \quad D = t H + r p_r. \quad (55)$$

Note that setting $n = 2$, $\mu_1 = \sin \theta$, $p_{\mu_1} = \frac{p_\theta}{\cos \theta}$, with (θ, p_θ) being a canonical pair, one reproduces the Hamiltonian of the conformal mechanics in four dimensions (14).

Computing the Casimir element of $so(2, 1)$ one gets the Hamiltonian of a reduced integrable spherical mechanics

$$\tilde{\mathcal{H}} = \frac{1}{(2n-3)(2n-2)} \sum_{i,j=1}^{n-1} ((2n-3) \rho_0^2 \delta_{ij} - \mu_i \mu_j) p_{\mu_i} p_{\mu_j} + \\ + \sum_{i,j=1}^{n-1} \left(\frac{(2n-3)(2n-2) \rho_0^2}{4 \mu_i^2} \delta_{ij} - \frac{(2n-3)^2 \rho_0^2}{4} - 1 \right) p_{\phi_i} p_{\phi_j} + m^2 \rho_0^2, \quad (56)$$

where ρ_0^2 is given in (52) and m^2 is now treated as a coupling constant. By construction, it inherits $U(n-1)$ symmetry of the background. A further reduction occurs if one disregards

⁵In (54) we redefine the momenta $p_{\phi_i} \rightarrow -p_{\phi_i}$ so as to conform with the notation adopted for the Hamiltonian mechanics in Sec. 2.2.

the cyclic variables ϕ_i and sets p_{ϕ_i} to be constants in (56)

$$\begin{aligned}\tilde{\mathcal{H}}_{red} &= \frac{1}{(2n-3)(2n-2)} \sum_{i,j=1}^{n-1} ((2n-3)\rho_0^2\delta_{ij} - \mu_i\mu_j)p_{\mu_i}p_{\mu_j} + \sum_{i=1}^{n-1} \frac{g_i^2\rho_0^2}{\mu_i^2} + \nu \sum_{i=1}^{n-1} \mu_i^2, \\ \rho_0^2 &= \frac{2(n-1)}{2n-3} - \sum_{i=1}^{n-1} \mu_i^2.\end{aligned}\tag{57}$$

Here ν and g_i are coupling constants.

5. Extremal black holes in $d = 2n + 1$ and integrable models

5.1. Near horizon geometry of the extremal maximally symmetric $d = 2n + 1$ black hole

If the dimension of spacetime is odd, there are n azimuthal angular coordinates ϕ_i and respectively n rotation parameters a_i . A vacuum solution of the Einstein equations describing a black hole in $2n + 1$ dimensions, which rotates in n orthogonal spatial 2-planes, reads [24]

$$\begin{aligned}ds^2 &= dt^2 - \frac{U}{\Delta}dr^2 - \frac{2M}{U} \left(dt - \sum_{i=1}^n a_i \mu_i^2 d\phi_i \right)^2 - \sum_{i=1}^n (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2), \\ \Delta &= \frac{1}{r^2} \prod_{i=1}^n (r^2 + a_i^2) - 2M, \quad U = \sum_{i=1}^n \frac{\mu_i^2}{r^2 + a_i^2} \prod_{j=1}^n (r^2 + a_j^2).\end{aligned}\tag{58}$$

As before, the latitudinal coordinates μ_i are subject to the constraint

$$\sum_{i=1}^n \mu_i^2 = 1,\tag{59}$$

and it is assumed that all μ_i lie in the interval $[0, 1]$. The isometry group of (58) includes the time translation and n rotations acting on the azimuthal variables.

In what follows we focus on the extremal and maximally symmetric configuration, for which

$$a_i = a, \quad M = \frac{n^n r_0^{2n-2}}{2}, \quad a^2 = (n-1)r_0^2.\tag{60}$$

The last two conditions follow from the requirement that $\Delta(r)$ has a double zero at the horizon radius $r = r_0$. Note that in this case the rotational symmetry group is known to enhance from $U(1)^n$ to $U(n)$.

In order to construct the near horizon metric, we rewrite (58) (where all the rotation

parameters are set equal) in the form

$$\begin{aligned}
ds^2 &= \frac{\Delta}{U} \left(dt - a \sum_{i=1}^n \mu_i^2 d\phi_i \right)^2 - \frac{U}{\Delta} dr^2 - \frac{1}{r^2} \sum_{i=1}^n \mu_i^2 (adt - (r^2 + a^2)d\phi_i)^2 \\
&\quad - (r^2 + a^2) \sum_{i=1}^n d\mu_i^2 + \frac{a^2(r^2 + a^2)}{r^2} \sum_{i<j}^n \mu_i^2 \mu_j^2 (d\phi_i - d\phi_j)^2, \\
\Delta &= \frac{(r^2 + a^2)^n}{r^2} - 2M, \quad U = (r^2 + a^2)^{n-1}, \quad \mu_n^2 = 1 - \sum_{i=1}^{n-1} \mu_i^2,
\end{aligned} \tag{61}$$

and redefine the coordinates

$$r \rightarrow r_0 + \epsilon r_0 r, \quad t \rightarrow \frac{nr_0 t}{2(n-1)\epsilon}, \quad \phi_i \rightarrow \phi_i + \frac{r_0 t}{2a\epsilon}. \tag{62}$$

Then we take into account the relation

$$\lim_{r \rightarrow r_0} \frac{\Delta(r)}{(r - r_0)^2} = \frac{2(n-1)(nr_0^2)^{n-1}}{r_0^2}, \tag{63}$$

and send ϵ to zero. This yields

$$\begin{aligned}
ds^2 &= r^2 dt^2 - \frac{dr^2}{r^2} - 2n(n-1) \sum_{i=1}^n d\mu_i^2 - 2 \sum_{i=1}^n \mu_i^2 (r dt + d\phi_i)^2 + \\
&\quad + \frac{2(n-1)}{n} \sum_{i<j}^n \mu_i^2 \mu_j^2 (d\phi_i - d\phi_j)^2, \quad \mu_n^2 = 1 - \sum_{i=1}^{n-1} \mu_i^2,
\end{aligned} \tag{64}$$

where we discarded an overall factor of $\frac{r_0^2}{2(n-1)}$ and scaled the azimuthal variables as follows: $\frac{an}{r_0} \phi_i \rightarrow \phi_i$. It is straightforward to verify that (64) is a vacuum solution of the Einstein equations. Note that setting $n = 2$, $\mu_1 = \sin \theta$ and $\mu_2 = \cos \theta$, one reproduces the metric (28) with $a = b$ and an overall factor of $\rho_0^2 = 2a^2$ being discarded.

Like in all the instances considered above, one can easily establish the $SO(2, 1)$ invariance of the near horizon metric (64) and the reducibility of the near horizon Killing tensor, which for the case at hand just coincides with the AdS_2 metric.

5.2. Conformal mechanics in $d = 2n + 1$ and its integrable reductions

In order to construct the Hamiltonian of a conformal mechanics, which is associated with

the background geometry (64), we invert the metric

$$g^{mn}(x)\partial_n\partial_m = \frac{1}{r^2}\left(\frac{\partial}{\partial t}\right)^2 - r^2\left(\frac{\partial}{\partial r}\right)^2 - \frac{2}{r}\sum_{i=1}^n\frac{\partial}{\partial t}\frac{\partial}{\partial\phi_i} + \frac{1}{2n(n-1)}\sum_{i,j=1}^{n-1}(\mu_i\mu_j - \delta_{ij})\frac{\partial}{\partial\mu_i}\frac{\partial}{\partial\mu_j} \\ + \sum_{i,j=1}^n\left(\frac{(n+1)}{2} - \frac{n}{2\mu_i^2}\delta_{ij}\right)\frac{\partial}{\partial\phi_i}\frac{\partial}{\partial\phi_j}, \quad \mu_n^2 = 1 - \sum_{i=1}^{n-1}\mu_i^2, \quad (65)$$

where δ_{ij} is the Kronecker delta, and solve the mass shell condition $g^{nm}p_np_m = m^2$ for p_0 ⁶

$$p_0 = H = r \left(\left[m^2 + (rp_r)^2 + \frac{1}{2n(n-1)}\sum_{i,j=1}^{n-1}(\delta_{ij} - \mu_i\mu_j)p_{\mu_i}p_{\mu_j} \right. \right. \\ \left. \left. + \sum_{i,j=1}^n\left(\frac{n}{2\mu_i^2}\delta_{ij} - \frac{(n+1)}{2}\right)p_{\phi_i}p_{\phi_j} + \left(\sum_{i=1}^np_{\phi_i}\right)^2 \right]^{1/2} - \sum_{i=1}^np_{\phi_i} \right). \quad (66)$$

The generators of special conformal transformations and dilatations read

$$K = \frac{1}{r} \left(\left[m^2 + (rp_r)^2 + \frac{1}{2n(n-1)}\sum_{i,j=1}^{n-1}(\delta_{ij} - \mu_i\mu_j)p_{\mu_i}p_{\mu_j} + \right. \right. \\ \left. \left. + \sum_{i,j=1}^n\left(\frac{n}{2\mu_i^2}\delta_{ij} - \frac{(n+1)}{2}\right)p_{\phi_i}p_{\phi_j} + \left(\sum_{i=1}^np_{\phi_i}\right)^2 \right]^{1/2} + \sum_{i=1}^np_{\phi_i} \right) + \\ + t^2H + 2trp_r, \quad D = tH + rp_r. \quad (67)$$

The Hamiltonian of a reduced integrable spherical mechanics is determined by the Casimir element of $so(2, 1)$

$$\tilde{\mathcal{H}} = \frac{1}{2n(n-1)}\sum_{i,j=1}^{n-1}(\delta_{ij} - \mu_i\mu_j)p_{\mu_i}p_{\mu_j} + \sum_{i,j=1}^n\left(\frac{n}{2\mu_i^2}\delta_{ij} - \frac{(n+1)}{2}\right)p_{\phi_i}p_{\phi_j}. \quad (68)$$

By construction, it inherits $U(n)$ symmetry of the background. Ignoring the cyclic variables ϕ_i , one obtains a further reduction

$$\tilde{\mathcal{H}}_{red} = \frac{1}{2n(n-1)}\sum_{i,j=1}^{n-1}(\delta_{ij} - \mu_i\mu_j)p_{\mu_i}p_{\mu_j} + \sum_{i=1}^n\frac{g_i^2}{\mu_i^2}, \quad \mu_n^2 = 1 - \sum_{i=1}^{n-1}\mu_i^2, \quad (69)$$

⁶In (66) the momenta p_{ϕ_i} were redefined $p_{\phi_i} \rightarrow -p_{\phi_i}$ so as to conform to the notation adopted for the Hamiltonian mechanics in Sec. 3.2.

where g_i^2 are coupling constants. Note that setting $n = 2$, $\mu_1 = \sin \theta$ and $p_{\mu_1} = \frac{p_\theta}{\cos \theta}$, with (θ, p_θ) being a canonical pair, one reproduces the Pöschl–Teller Hamiltonian (40).

6. Conclusion

To summarize, in this work we have constructed a metric describing the near horizon geometry of an extremal rotating black hole in arbitrary dimension for the special case that all the rotation parameters are equal. The Hamiltonian of an integrable spherical mechanics associated with such a geometry was derived.

Many issues related to the present work deserve a further study. First of all, it is important to explicitly describe $U(n)$ symmetry underlying the models beyond five dimensions. This will require a parametrization of the sphere in the sector of latitudinal variables by spherical coordinates. As was demonstrated in Section 2 and Section 3, the most interesting models arise if one considers a further reduction of the spherical mechanics, which is obtained by discarding cyclic variables. It is interesting to describe these models in full generality and to uncover their superintegrability. In odd-dimensional space-time such a system can be interpreted as a particle moving on $(n - 1)$ -dimensional sphere and interacting with an external field. Geometry behind a similar model (57) corresponding to an even-dimensional space-time is of considerable interest. A possible relation between the models constructed in this work and A_n -models in the classification of Olshanetsky and Perelomov [29] is worth studying.

In this work we have considered a maximally symmetric black hole configuration, which in general yields a superintegrable spherical mechanics. Less symmetric configurations, for which not all the rotation parameters are equal, will result in a spherical mechanics with less symmetry, which, however, may maintain the property of integrability. It would be interesting to derive a constraint on the rotation parameters of a black hole, which guarantees that the resulting spherical mechanics is integrable. Another interesting issue is a generalization of the present analysis to the case of the near horizon extremal rotating black hole on AdS background. A nonvanishing cosmological constant will yield a more complicated potential for the spherical mechanics.

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